Complete tripartite graphs and their competition numbers

Jaromy Kuhl*

Abstract

We present a piecewise formula for the competition numbers of the complete tripartite graphs. For positive integers x, y and z where $2 \leq x \leq y \leq z$, the competition number of the complete tripartite graph $K_{x,y,z}$ is $yz - z - y - x + 3$ whenever $x \neq y$ and $yz - 2y - z + 4$ otherwise.

1 Introduction

In this note we consider competition graphs as introduced by Cohen in [1] and we consider a problem left open by Kim and Sano in [3]. Let D be a digraph with vertex set V and arc set A. If $u, v \in V$ have a common out-neighbor in D, then u and v are said to be in competition. The simple graph (V, E) in which edge set E is defined as

$$E = \{\{u, v\} : \ u \text{ and } v \text{ are in competition in } D\}$$

is called the competition graph of D and is denoted $C(D)$. Given the applicative nature of competition graphs (one example is that V represents a set of organisms in a food-web and competition is defined by organisms competing for food), it is important to ask which graphs are competition graphs of acyclic digraphs. In [8], Roberts observed that for any graph G and for a sufficiently large integer k, $G \cup I_k$ is the competition graph of an acyclic digraph, where I_k denotes the graph on k isolated vertices. The minimum such k is called the competition number of G. Formally, the competition number of G is

$$k(G) = \min\{k : \ G \cup I_k = C(D) \text{ in which } D \text{ is an acyclic digraph}\}.$$

In general, the problem of computing $k(G)$ is NP-hard [5]. So to reduce generality, G will belong to the class of complete multipartite graphs. The following theorems are what is currently known concerning the competition numbers of compete multipartite graphs.

*University of West Florida, Department of Mathematics and Statistics, Pensacola, Fl 32514; jkuhl@uwf.edu

Theorem 1.1 *The competition number of the complete bipartite graph K_{n_1,n_2} is $n_1n_2 - n_1 - n_2 + 2$.*

Theorem 1.1 is a corollary of the statement that if G is a triangle-free connected graph, then $k(G) = |E(G)| - |V(G)| + 2$. Recently, Kim and Sano [3] found the competition number of the complete tripartite graph $K_{n,n,n}$.

Theorem 1.2 *The competition number $k(K_n^3)$ is $n^2 - 3n + 4$.*

We extend Kim and Sano's result to complete tripartite graphs in which the partite sets may not have equal size. We prove the following formula:

Theorem 1.3 *For positive integers x, y and z where $2 \leq x \leq y \leq z$,*

$$k(K_{x,y,z}) = \begin{cases} yz - 2y - z + 4, & \text{if } x = y \\ yz - z - y - x + 3, & \text{if } x \neq y \end{cases}$$

Some progress has been made on competition numbers of the complete tetrapartite graph K_n^4 [4] and, more generally, the complete multipartite graph K_n^m [4].

Theorem 1.4 *If $n \geq 5$ is odd, then*

$$n^2 - 4n + 7 \leq k(K_n^4) \leq n^2 - 4n + 8.$$

Theorem 1.5 *If n is prime and $m \leq n$, then*

$$k(K_n^m) \leq n^2 - 2n + 3.$$

Park et al. [7] give bounds for the general case with respect to $L(n)$, the largest size of a family of mutually orthogonal latin squares of order n.

Theorem 1.6 *If m and n are positive integers such that $3 \leq m \leq L(n) + 2$, then*

$$k(K_n^m) \leq n^2 - n + 1.$$

For small values of n, Park et al. [6] found the following competition numbers.

Theorem 1.7 *If $m \geq 2$, then $k(K_2^m) = 2$ and if $m \geq 3$, then $k(K_3^m) = 4$.*

While we do not do so in this paper, it would be interesting to study the competition number of K_{n_1,n_2,n_3,n_4} since very little is currently known. Furthermore, there remains much to be known on computing the competition number $k(K_n^m)$.

2 Edge clique covers of $K_{x,y,z}$

Let $U = \{u_1, \ldots, u_x\}$, $V = \{v_1, \ldots, v_y\}$, and $W = \{w_1, \ldots, w_z\}$ be the vertex partition sets of $K_{x,y,z}$ where $2 \le x \le y \le z$. We use $\Delta(i, j, k)$ to denote the clique induced on the vertex set $\{u_i, v_j, w_k\}$ and we use $\Delta(j, k)$ to denote the clique induced on the vertex set $\{v_j, w_k\}$. Note that a clique of order 3 is the largest clique in $K_{x,y,z}$.

Competition numbers can be computed by first finding a minimal edge clique cover. Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a family of cliques in a graph G; i.e. the subgraph induced on $S_i \subseteq V(G)$ is complete for each $i \in [m]$. The family $\mathcal{S}$ is called an edge clique cover of G provided $\{u, v\} \in E(G)$ if and only if $\{u, v\} \subseteq S_i$ for some $i \in [m]$. The edge clique cover number of G, denoted $\theta_e(G)$, is

$$\theta_e(G) = \min\{|\mathcal{S}| : \ \mathcal{S} \text{ is an edge clique cover of } G\}.$$

Certainly, for any graph G, $k(G) \le \theta_e(G)$. Indeed, if $\theta_e(G) = k$, then each vertex of a clique in G can be directed to a vertex of I_k in the digraph D.

We find a minimal edge clique cover of $K_{x,y,z}$ using r-semi latin squares. An r-semi latin square of order n is an $n \times n$ array such that each element (or symbol) from the set $S = \{s_1, s_2, \ldots, s_{nr}\}$ appears in each row and each column, and each cell contains r elements. If we label the rows and columns with sets $R = \{r_1, r_2, \ldots, r_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$ respectively, we may think of an r-semi latin square as a set of ordered triples (r_i, c_j, s_k), where symbol s_k appears at the intersection of row r_i and column c_j. Where convenient, we use the notation $c_j \circ s_k$ to denote the row containing symbol s_k in column c_j.

Henceforth q and r are positive integers such that $z = qy + r$, where $0 \le r < y$. Let L be a $(q+1)$-semi latin square of order y on the symbol set $S = \{s_1, \ldots, s_{(q+1)y}\}$. Furthermore, let $R' = \{r'_1, \ldots, r'_x\} \subseteq R$ be a set of x rows and let $S' = \{s'_1, \ldots, s'_z\} \subseteq S$ be a set of z symbols. We use

$$L(R', C, S') = \{(r'_i, c_j, s'_k) : \ (r'_i, c_j, s'_k) \in L, \ r'_i \in R', \ s'_k \in S'\}$$

to denote the $x \times y$ array on symbol set S' induced by the intersection of rows R' and columns C. Note that the family $\mathcal{F}$, defined below, is a subset of an edge clique cover of $K_{x,y,z}$. In fact, we will later show that $\mathcal{F}$ is a minimal edge clique cover of $K_{x,y,z}$.

$$\mathcal{F} = \{\Delta(i, j, k) : \ (r'_i, c_j, s'_k) \in L(R', C, S')\} \cup$$

$$\{\Delta(j, k) : \ (c_j \circ s_k, c_j, s_k) \in L(R \setminus R', C, S')\} \tag{1}$$

For an example of (1), consider $K_{2,4,6}$. Since $z = 6$ and $y = 4$, $q = 1$. We use the following 2-semi latin square of order 4 as L and set $R' = \{r_1, r_4\}$ and $S' = \{s_1, \ldots, s_6\}$, where $r'_1 = r_1$, $r'_2 = r_4$ and $s'_i = s_i = i$ for $1 \le i \le 6$.

1,2	4,5	3,7	6,8
5,6	7,8	1,2	3,4
7,8	2,3	4,6	1,5
3,4	1,6	5,8	2,7

Then the rectangular array $L(R', C, S')$ is

1,2	4,5	3	6
3,4	1,6	5	2

The clique $\Delta(1,1,2)$ is included in $\mathcal{F}$ since $(r'_1, c_1, s'_2) \in L(R', C, S')$. The same can be said of $\Delta(2,1,3)$ since $(r'_2, c_1, s'_3) \in L(R', C, S')$. Also, since $(r_2, c_1, s'_5) \in L(R \setminus R', C, S')$, $\Delta(1,5) \in \mathcal{F}$. The remaining members of $\mathcal{F}$ are given in the following family;

$$\begin{aligned}\mathcal{F} = \{&\Delta(1,1,1), \Delta(1,1,2), \Delta(1,2,4), \Delta(1,2,5), \Delta(1,3,3), \Delta(1,4,6),\\ &\Delta(2,1,3), \Delta(2,1,4), \Delta(2,2,1), \Delta(2,2,6), \Delta(2,3,5), \Delta(2,4,2),\\ &\Delta(1,5), \Delta(1,6), \Delta(3,1), \Delta(3,2), \Delta(4,3), \Delta(4,4), \Delta(2,2),\\ &\Delta(2,3), \Delta(3,4), \Delta(3,6), \Delta(4,1), \Delta(4,5)\}\end{aligned}$$

Lemma 2.1 *The family $\mathcal{F}$ is an edge clique cover of $K_{x,y,z}$. Moreover, $\mathcal{F}$ is minimal and $\theta_e(K_{x,y,z}) = yz$.*

Proof: First, we show that $\mathcal{F}$ is an edge clique cover of $K_{x,y,z}$. Let $R' = \{r'_1, \ldots, r'_x\} \subseteq R$ be a set of x rows and let $S' = \{s'_1, \ldots, s'_z\}$ be a set of z symbols in a $(q+1)$-semi latin square L of order y. Consider the edge $e = \{u_i, v_j\}$ in $K_{x,y,z}$, $i \in [x]$ and $j \in [y]$. Let $S_{i,j}$ denote the set of $q+1$ symbols at the intersection of r'_i and c_j. If $S_{i,j} \cap S' = \emptyset$, then $q+1 \leq q-r$, a contradiction as $r \geq 0$. Therefore there is an integer k such that $(r'_i, c_j, s'_k) \in L(R', C, S')$. Thus the clique $\Delta(i,j,k) \in \mathcal{F}$ covers the edge e.

Now set $e = \{u_i, w_j\}$, $i \in [x]$ and $j \in [z]$. Since each symbol of S' appears in each row of $L(R', C, S')$, there is an integer k such that $(r'_i, c_k, s'_j) \in L(R', C, S')$. Hence $\Delta(i,k,j) \in \mathcal{F}$ covers e. Finally, set $e = \{v_i, w_j\}$, $i \in [y]$ and $j \in [z]$. There is an integer $k \in [y]$ so that $r_k = c_i \circ s'_j$. If $r_k \in R'$, then certainly e is covered by a clique of order three in $\mathcal{F}$. Otherwise $r_k \in R \setminus R'$ and $\Delta(i,j)$ covers e.

We finish the proof by showing that yz is a lower and upper bound for $\theta_e(K_{x,y,z})$. Note that there are yz edges of the form $\{v, w\}$ where $v \in V$ and $w \in W$. Furthermore, there is no clique in $K_{x,y,z}$ that contains two edges of the form $\{v, w\}$. It follows that at least yz cliques are needed to cover the edges that contain end vertices in partitions V and W. Hence $\theta_e(K_{x,y,z}) \geq yz$. To show that yz is an upper bound for $\theta_e(K_{x,y,z})$, we need only to provide an edge clique cover of $K_{x,y,z}$ whose cardinality is yz. From above, $\mathcal{F}$ is an edge clique cover of $K_{x,y,z}$. Since L contains precisely $y^2(q+1)$ triples and since symbols from $S \setminus S'$ appear precisely y times in L, $\mathcal{F}$ is made of

$$y^2(q+1) - y(y(q+1) - z) = yz$$

triples. Hence $\theta_e(K_{x,y,z}) \leq yz$. Moreover, this shows that $\mathcal{F}$ is a minimal edge clique cover of $K_{x,y,z}$. □

To end this section we comment on a general minimal edge clique cover of $K_{x,y,z}$ when $x = y$.

Lemma 2.2 *Let $\mathcal{S}$ be a minimal edge clique cover of $K_{y,y,z}$ and let $S, S' \in \mathcal{S}$. If $|S \cap S'| = 2$, then $S = \{u, v, w\}$ and $S' = \{u, v, w'\}$ where $u \in U$, $v \in V$ and $w, w' \in W$.*

Proof: We may assume that $|S| = 3$ for each $S \in \mathcal{S}$. Let $S, S' \in \mathcal{S}$ such that $|S \cap S'| = 2$. Since $|\mathcal{S}| = yz$ (by Lemma 2.1) and since no clique contains two edges of the form $\{v, w\}$, no two cliques of $\mathcal{S}$ share an edge of the form $\{v, w\}$ where $v \in V$ and $w \in W$. Similarly, no two cliques share an edge of the form $\{u, w\}$ where $u \in U$ and $w \in W$. Hence $(S \cup S') \setminus (S \cap S') \subseteq W$. □

3 Proof of main result

The following characterization of competition graphs [2] is used to show a lower bound for $k(K_{x,y,z})$.

Theorem 3.1 *A graph G is the competition graph of an acyclic digraph if and only if there exists an ordering $a_1, \ldots, a_n$ of the vertices of G and an edge clique cover $\{S_1, \ldots, S_n\}$ of G such that if $a_i \in S_j$, then $i < j$.*

An equivalent way of stating Theorem 3.1 is to say that there exists an ordering $a_1, \ldots, a_n$ of the vertices of G and an edge clique cover $\{S_1, \ldots, S_n\}$ of G such that $S_i \subseteq \{a_1, \ldots, a_{i-1}\}$ for each i.

Theorem 3.2 *For integers x, y and z where $2 \leq x \leq y \leq z$,*

$$k(K_{x,y,z}) \geq yz - z - y - x + 3.$$

Moreover, if $x = y$, then

$$k(K_{y,y,z}) \geq yz - 2y - z + 4.$$

Proof: Let $k = k(K_{x,y,z})$ and let D denote an acyclic digraph such that $C(D) = K_{x,y,z} \cup I_k$. Note that $\mathcal{S}$ is an edge clique cover of $K_{x,y,z}$ if and only if $\mathcal{S}$ is an edge clique cover of $K_{x,y,z} \cup I_k$. Then, from Theorem 3.1, there is an ordering $a_1, \ldots, a_{x+y+z+k}$ of the vertices of $K_{x,y,z} \cup I_k$ and an edge clique cover $\mathcal{S} = \{S_1, \ldots, S_{x+y+z+k}\}$ of $K_{x,y,z}$ such that $S_i \subseteq \{a_1, \ldots, a_{i-1}\}$ for each i. We may assume that the order of each non empty clique in $\mathcal{S}$ is three. Then $S_1 = S_2 = S_3 = \emptyset$ and so, by Lemma 2.1, $|S \setminus \{S_1, S_2, S_3\}| \geq yz$. Hence $x + y + z + k - 3 \geq yz$ and so $k \geq yz - x - y - z + 3$.

Suppose now that $x = y$ and that, for the sake of contradiction, $k = yz-2y-z+3$. Then S_i is non empty for each $i \geq 4$, $S_4 = \{a_1, a_2, a_3\}$ and $S_5 \subset \{a_1, a_2, a_3, a_4\}$. So it must be that $|S_4 \cap S_5| = 2$. Without loss of generality, assume that $S_5 = \{a_2, a_3, a_4\}$. By Lemma 2.2, $a_1, a_4 \in W$. Let $l \geq 4$ be the largest integer such that $S_{l+1} = \{a_2, a_3, a_l\}$ and $a_l \in W$. Then $S_{l+2} = \{a_2, a_j, a_{l+1}\}$ or $S_{l+2} = \{a_3, a_j, a_{l+1}\}$, $j \in [l] \setminus \{2, 3\}$. In either case $|S_{l+2} \cap S_1| = 2$ or $|S_{l+2} \cap S_{j+1}| = 2$. But $a_{l+1} \in U \cup V$, contradicting Lemma 2.2. Hence $k \geq yz - 2y - z + 4$. $\square$

We now proceed to the main result. Henceforth L is a $(q+1)$-semi latin square of order y such that $(r_i, c_j, s_k) \in L$ if and only if $i + j - 1 \equiv k \mod y$. Furthermore, we set $R' = \{r_1, \ldots, r_{x-1}, r_y\}$ and $S' = \{s_1, \ldots, s_z\}$ where $r'_i = r_i$ for $i \in [x-1]$, $r'_x = r_y$ and $s'_i = s_i$ for $i \in [z]$. For $y = 5$ and $z = 13$, the arrays below are L and $L(R', C, S')$ respectively.

1,6,11	2,7,12	3,8,13	4,9,14	5,10,15
2,7,12	3,8,13	4,9,14	5,10,15	1,6,11
3,8,13	4,9,14	5,10,15	1,6,11	2,7,12
4,9,14	5,10,15	1,6,11	2,7,12	3,8,13
5,10,15	1,6,11	2,7,12	3,8,13	4,9,14

1,6,11	2,7,12	3,8,13	4,9	5,10
2,7,12	3,8,13	4,9	5,10	1,6,11
5,10	1,6,11	2,7,12	3,8,13	4,9

Proof of Theorem 1.3. Case 1: $x = y$.

In this case $r'_i = r_i$ for each i. We first order the vertices $a_1, \ldots, a_{2y+z}$ of $K_{y,y,z}$ as

$$u_1, v_1, w_1, u_2, v_y, w_y, u_y, v_2, w_2, u_{y-1}, v_{y-1}, w_{y-1}, \ldots, u_3, v_3, w_3, w_{y+1}, \ldots, w_z.$$

Note that all vertices of $K_{y,y,z}$ appear in the vertex ordering. Next, we order $2y + z - 3$ cliques of $\mathcal{F}$ in the following way. The first 6 cliques are ordered as

$$\Delta_1 = \{u_1, v_1, w_1\},\ \Delta_2 = \{u_2, v_y, w_1\},\ \Delta_3 = \{u_1, v_y, w_y\},$$

$$\Delta_4 = \{u_y, v_1, w_y\},\ \Delta_5 = \{u_y, v_2, w_1\},\ \Delta_6 = \{u_1, v_2, w_2\}$$

For $0 \leq s \leq y-4$, the next $3y - 9$ cliques are given as

$$\begin{aligned}\Delta_{3s+7} &= \{u_{y-s-1}, v_2, w_{y-s}\},\\ \Delta_{3s+8} &= \{u_2, v_{y-s-1}, w_{y-s}\}, \text{ and}\\ \Delta_{3s+9} &= \{u_1, v_{y-s-1}, w_{y-s-1}\}.\end{aligned}$$

Finally, for $0 \leq s \leq z-y-1$, the remaining $z-y$ cliques are

$$\Delta_{3y-2+s} = \{u, v, w_{y+s+1}\},$$

where $u \in U$ and $v \in V$ are any vertices such that $\{u, v, w_{y+s+1}\} \in \mathcal{F}$.

Given the vertex and clique orderings above, we construct a digraph that shows that $yz - 2y - z + 4$ is an upper bound for $k(K_{y,y,z})$. We must first note that

$$\Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_i \subseteq \{a_1, a_2, \ldots, a_{i+3}\} \tag{2}$$

for $i \in [2y+z-3]$. This follows from the fact that $\Delta_1 = \{a_1, a_2, a_3\}$, $\Delta_2 \setminus \Delta_1 = \{a_4, a_5\}$ and $\Delta_i \setminus (\Delta_1 \cup \ldots \cup \Delta_{i-1}) = \{a_{i+3}\}$, where $3 \leq i \leq 2y+z-3$.

Since $\mathcal{F}$ is a minimal edge clique cover, there are $yz - 2y - z + 3$ cliques in $\mathcal{F} \setminus \{\Delta_1, \ldots, \Delta_{2y+z-3}\}$. Set $\mathcal{F} \setminus \{\Delta_1, \ldots, \Delta_{2y+z-3}\} = \{T_1, \ldots, T_{yz-2y-z+3}\}$. Let D be the digraph with the following vertex set V and arc set A;

$$\begin{aligned} V(D) =& \{a_1, \ldots, a_{z+2y-3}\} \cup \{\alpha_0, \ldots, \alpha_{yz-2y-z+3}\} \\ A(D) =& \bigcup_{i=1}^{2y+z-4} \{(\delta, a_{i+4}) : \ \delta \in \Delta_i\} \cup \{(\delta, \alpha_0) : \ \delta \in \Delta_{2y+z-3}\} \cup \\ & \bigcup_{i=1}^{yz-2y-z+3} \{(\delta, \alpha_i) : \ \delta \in T_i\}. \end{aligned}$$

From statement (2), the digraph D is acyclic. Because every clique in $\mathcal{F}$ has a common out-neighbor in D, $E(C(D)) \subseteq E(K_{y,y,z})$. Moreover, the in-nighborhood of a vertex of D is a clique in $\mathcal{F}$. Therefore $E(K_{y,y,z}) \subseteq E(K_{y,y,z})$. It follows that $C(D) = K_{y,y,z} \cup I_{yz-2y-z+4}$. Hence, by Theorem 3.2,

$$k(K_{y,y,z}) = yz - 2y - z + 4.$$

Case 2: $x < y$.

In this case $r'_i = r_i$ for each $i \in [x-1]$ and $r'_x = r_y$. As in the previous case, we begin by ordering vertices and cliques in $K_{x,y,z}$. The vertex ordering $a_1, \ldots, a_{x+y+z}$ of $K_{x,y,z}$ is

$$u_x, v_1, w_y, v_2, w_1, u_1, w_2, u_2, v_y, w_{y-1}, v_{y-1}, \ldots, w_x, v_x,$$

$$u_{x-1}, v_{x-1}, w_{x-1}, \ldots, u_3, v_3, w_3, w_{y+1}, \ldots, w_z.$$

Next, we order $x+y+z-2$ cliques of $\mathcal{F}$. The first 7 cliques are ordered as

$$\Delta_1 = \{u_x, v_1, w_y\}, \quad \Delta_2 = \{v_2, w_y\}, \quad \Delta_3 = \{u_x, v_2, w_1\},$$
$$\Delta_4 = \{u_1, v_1, w_1\}, \quad \Delta_5 = \{u_1, v_2, w_2\}, \quad \Delta_6 = \{u_2, v_1, w_2\}, \quad \Delta_7 = \{u_2, v_y, w_1\}$$

For $0 \leq s \leq y-x-1$, the next $2(y-x)$ cliques in the ordering are given as

$$\begin{aligned} \Delta_{2s+8} &= \{u_x, v_{y-s}, w_{y-s-1}\}, \text{ and} \\ \Delta_{2s+9} &= \{u_1, v_{y-s-1}, w_{y-s-1}\}. \end{aligned}$$

For $0 \le s \le x-4$, the next $3x-9$ cliques are given as

$$\Delta_{3s+2(y-x)+8} = \{u_{x-s-1}, v_2, w_{x-s}\},$$
$$\Delta_{3s+2(y-x)+9} = \{u_2, v_{x-s-1}, w_{x-s}\}, \text{ and}$$
$$\Delta_{3s+2(y-x)+10} = \{u_1, v_{x-s-1}, w_{x-s-1}\}.$$

Finally, for $0 \le s \le z-y-1$, the remaining $z-y$ cliques in the ordering are

$$\Delta_{2y+x-1+s} = \{u, v, w_{y+s+1}\},$$

where $u \in U$ and $v \in V$ are any vertices such that $\{u, v, w_{y+s+1}\} \in \mathcal{F}$.

In this case note that

$$\Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_i \subseteq \{a_1, a_2, \ldots, a_{i+2}\} \tag{3}$$

for $i \in [z+y+x-4]$. Set $\mathcal{F}' \setminus \{\Delta_1, \ldots, \Delta_{z+y+x-4}\} = \{T_1, \ldots, T_{yz-z-y-x+4}\}$. Let D be the digraph with the following vertex set V and arc set A;

$$V(D) = \{a_1, \ldots, a_{z+y+x-4}\} \cup \{\alpha_0, \ldots, \alpha_{yz-z-y-x+2}\}$$
$$A(D) = \bigcup_{i=1}^{z+y+x-3} \{(\delta, a_{i+3}) : \ \delta \in \Delta_i\} \cup \{(\delta, \alpha_0) : \ \delta \in \Delta_{z+y+x-4}\} \cup \bigcup_{i=1}^{yz-z-y-x+2} \{(\delta, \alpha_i) : \ \delta \in T_i\}.$$

It follows from statement (3) that D is acyclic. Furthermore $C(D) = K_{x,y,z} \cup I_{yz-z-y-x+3}$. Hence, by Theorem 3.2,

$$k(K_{x,y,z}) = yz - z - y - x + 3.$$

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