On the Existence of Transversals in *r*-Multi Latin Squares

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Abstract

In this paper, we will investigate the existence of transversals in r-multi Latin squares, a generalization of the Latin square. We will reveal values of r that guarantee the existence of a transversal. In addition, we will also generalize David E. Woolbright's proof (which is specific to Latin squares) to determine a lower bound on the size of a maximal transversal in any r-multi Latin square. We will also touch upon the existence of transversals on any given set of n symbols. Finally, we will present many suggestions for future research regarding this intriguing mathematical construct.

I. INTRODUCTION

A. Statement of Problem

We begin stating our problem of interest by defining important terms.

1. Definitions of Important Terms

A **Latin square of order *n*** is an $n \times n$ array of n symbols where each symbol appears exactly once in each row and once in each column. Below is an example of a Latin square of order 5.

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

A Latin square of order n can also be defined as a set of n^2 ordered triples where the triple (i, j, s) refers to row i, column j, and symbol s. If we label each unique row, column, and symbol with an integer from 1 to n, any Latin square

$L = \{(i, j, s) \mid 1 \le i, j, s \le n\}$ has the following three properties:

(1) There is exactly 1 occurrence of each unique ordered pair (i, j);

(2) There is exactly 1 occurrence of each unique ordered pair (i, s);

(3) There is exactly 1 occurrence of each unique ordered pair (j, s).

A Latin square of order *n* is also equivalent to a proper *n*-coloring of the edges of the complete bipartite graph $K_{n,n}$. To show this, we begin by creating $K_{n,n}$ with the 2 partitions of vertices $A = \{a_1, a_2, ..., a_{n-1}, a_n\}$ and $B = \{b_1, b_2, ..., b_{n-1}, b_n\}$. Next, create a 1 to 1 mapping from each unique symbol in the Latin square to a unique color. We do this by assigning each of the *n* unique symbols in the Latin square with a unique integer from 1 to *n*. Then, we let $\{c_1, c_2, ..., c_{n-1}, c_n\}$ be *n* distinct colors. Finally, we color edge (a_i, b_j) in $K_{n,n}$ with color c_k if and only if cell (i, j) contains symbol *k* in the Latin square. *Figure 1* below illustrates this equivalence when $n = 4$. Let the colors Red, Green, Blue and Yellow represent the symbols R, G, B, and Y, respectively.

R	G	B	Y
G	B	Y	R
B	Y	R	G
Y	R	G	B

$\equiv$

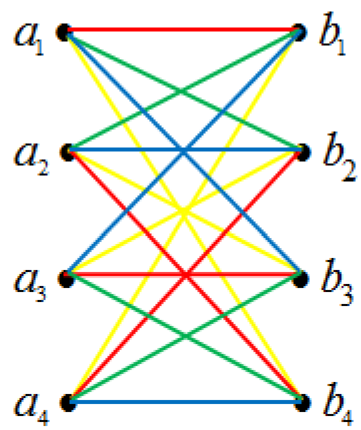

Figure 1. A proper 4-coloring representation of a 4 × 4 Latin square.

An ***r*-multi Latin square of order *n***, a generalization of the Latin square, is an *n* × *n* array of nr symbols where each entry (or cell) contains a set of *r* symbols and each symbol appears exactly once in each row and once in each column [7]. Thus, a 1-multi Latin square of order *n* is also a Latin square of order *n*.

Here is an example of a 3-multi Latin square of order 5.

1 2 3	4 5 6	7 8 9	10 11 12	13 14 15
7 8 10	1 2 9	13 14 15	3 4 5	6 11 12
4 5 9	3 7 8	6 11 12	13 14 15	1 2 10
12 14 15	10 11 13	1 2 3	6 8 9	4 5 7
6 11 13	12 14 15	4 5 10	1 2 7	3 8 9

The *r*-multi Latin square can also be defined as a set of n^2r ordered triples where the triple (i, j, s) refers to row *i*, column *j*, and symbol *s*. If we label each unique symbol with an integer from 1 to *nr*, any *r*-multi Latin square

$$R = \{(i, j, s) \mid 1 \le i, j \le n \;;\; 1 \le s \le nr\}$$ has the following three properties:

(1) There are exactly *r* distinct symbols tripled with each unique ordered pair (i, j);

(2) There is exactly 1 occurrence of each unique ordered pair (i, s);

(3) There is exactly 1 occurrence of each unique ordered pair (j, s).

A **partial transversal** of an *r*-multi Latin square is a subset of the n^2r ordered triples (that make up the *r*-multi Latin square) such that no 2 ordered triples have the same row, column, or symbol. That is, any partial transversal

$$P = \{(i, j, s) \mid 1 \le i, j \le n \;;\; 1 \le s \le nr\}$$ has the following three properties:

(1) There is exactly 1 occurrence of each row *i*;

(2) There is exactly 1 occurrence of each column *j*;

(3) There is exactly 1 occurrence of each symbol *s*.

The length (or size) of a partial transversal is the number of ordered triples (or pairwise distinct symbols) in the set. Below is an example of a partial transversal of length 3 in a 1-multi Latin square of order 4.

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

If the length *k* of the partial transversal is maximized to the order of the Latin square where $k = n$, then the partial transversal is called a **transversal**. Note that the example above contains no transversals. We will refer to any collection of *n* cells that meet requirements 1 and 2 above as a **diagonal collection**. In other words, a diagonal collection of cells is a potential transversal in that the row and column requirements are met, but not necessarily the symbol requirement. For example, the main diagonal (A, C, A, C) in the 1-multi Latin square of order 4 above is a diagonal collection. Highlighted below is an example of a transversal (as well as a diagonal collection) in a 1-multi Latin square of order 5.

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

Highlighted below is an example of a transversal in a 3-multi Latin square of order 5.

1 2 3	4 5 6	7 8 9	10 11 12	13 14 15
7 8 10	1 2 9	13 14 15	3 4 5	6 11 12
4 5 9	3 7 8	6 11 12	13 14 15	1 2 10
12 14 15	10 11 13	1 2 3	6 8 9	4 5 7
6 11 13	12 14 15	4 5 10	1 2 7	3 8 9

It seems somewhat intuitive that a transversal is more likely to exist as the number of symbols per cell, which is *r*, is increased. As shown in the example above, we may have multiple occurrences of symbols (i.e. 1, 2, 7, and 8) among a diagonal collection of cells and yet still have enough variety to obtain a complete transversal. Of course, the increased likelihood of the existence of a transversal is easier speculated than proven.

Much research has been done to establish a lower bound for the maximum length of a partial transversal in any Latin square of order *n*. Thus far, every proven lower bound is significantly less than two widely known and undisputed conjectured values. These are:

Brualdi's Conjecture: Every Latin square of order n contains a partial transversal of length at least $n-1$.

Ryser's Conjecture: Every Latin square of odd order contains a transversal.

In 1978, one of the more elegant results was shown by David E. Woolbright [14]. In this proof, Woolbright used the proper *n*-coloring of the complete bipartite graph representation of the Latin square to show that every Latin square of order n contains a partial transversal of size at least $n-\sqrt{n}$. Of course as *n* gets large, this result gets significantly less than the above conjectured values of *n* (for odd *n*) and $n-1$ (for even *n*).

2. Questions to Answer

Once research began, it became obvious that there are many unanswered questions about transversals in Latin squares, and even more so about transversals in *r*-multi Latin squares. In addition, the act of researching itself led the way to even more questions which are presented in the *Suggestions for Further Study* section. In this paper, we will attempt to answer the following questions.

1. How large does r need to be in order to guarantee the existence of a transversal in any r-multi Latin square of order n?
2. Can we generalize Woolbright's proof to establish a lower bound for the maximum length of a partial transversal in any r-multi Latin square?
3. Can we find a transversal on n specified symbols if r is big enough?

B. Relevance of Problem

For over 3000 years, the Latin square construct has been utilized in a variety of ways. Its first known use was on amulets during the medieval Islamic time period to display letters from a name of God. The first known written application of the Latin square in 1723 is known as the 16 card trick [8]. Here, we must arrange all of the face cards (aces, jacks, kings, and queens) in a 4×4 array such that each column and each row contains 1 card from each suit (clubs, diamonds, hearts, and spades) and 1 card from each face. Amazingly, there are 6,912 ways to solve the puzzle and 20,922,789,881,088 ways to fail to solve the puzzle [11]. That is only about 33 billionths of a percent! Thus by pure guesswork, one has about a 17 times better chance of winning the Florida Powerball Lottery at about 571 billionths of a percent [14]. Here is a solution to the 16 card trick [11]:

♠A	♥K	♦J	♣Q
♥Q	♠J	♣K	♦A
♣J	♦Q	♥A	♠K
♦K	♣A	♠Q	♥J

The solution is actually a specific type of r-multi Latin square where no pair of symbols occurs more than once in each cell. This is called a **Simple Orthogonal Multi-Array** (or **SOMA**) [3]. Note that the arrangement above is also the superposition of 2 Latin Squares formed by the faces and the suits. When the superposition of 2 Latin squares contains the n^2 unique combinations of symbols, the 2 Latin squares are said to be **orthogonal** (or Graeco-Latin). Leonhard Euler defined this concept around 1779 while trying to solve the legendary “Problem of the 36 Officers.” [8] Similar to the problem above, here we wish to arrange 36 officers in a 6 × 6 array such that each column and each row contains 1 officer from 6 different ranks and 1 officer from 6 different regiments. It was not until 1901 when Gaston Terry showed by enumeration that 2 orthogonal Latin Squares of order 6 do not exist. Thus, the Problem of the 36 Officers has no solution. In fact, it has been proved that orthogonal Latin squares of every order exist except for $n = 2$ and $n = 6$.

Orthogonal Latin squares inspired interest in transversals [10]. One interesting theorem states that a Latin square has an orthogonal mate if and only if the Latin square can be decomposed into disjoint transversals. This means that the Latin square contains n distinct transversals with no entries in common among the transversals. There are many published research papers that investigate transversals in Latin squares. There are even publications that explore a generalization of the transversal called a **k-plex** [9]. The generalization of the Latin square called an r-multi Latin square is a relatively new construct in the combinatorial sense. However, there is an identical construct referred to as a semi-Latin square that was used extensively in statistical applications by a number of statisticians including R.A Fisher [1]. As demonstrated above, exploring the nature of transversals in r-multi Latin squares is indeed an intriguing, fruitful, and relevant area of research.

C. Literature Review

Numerous papers have been published regarding transversals in Latin squares, but none (as far as this author has discovered) regarding transversals in *r*-multi Latin squares. A major question that has remained unanswered for over 40 years is "What is the upper lower bound for the maximum length of a partial transversal in any given Latin square?" As mentioned earlier, every proven lower bound is significantly less than Brualdi's and Ryser's two widely known and undisputed conjectured values. Both conjectures have been confirmed for orders 11 and below by enumeration [11]. Among the most popular proven lower bounds are found below in Table 1.

Author(s)	**Proven Lower Bound**	**Date**
Hatami and Shor [5]	$n - 11.053\log^2 n$	1982
Woolbright [14]	$n - \sqrt{n}$	1978
Brouwer, Vries, and Wieringa [2]	$n - \sqrt{n}$	1978
Wang [12]	$\frac{9}{11}n$	1978
Drake [4]	$\frac{3}{4}n$	1977
Koksma [6]	$\frac{2}{3}n$	1969

Table 1. Lower Bounds for Maximal Partial Transversal Length in a Latin square of order *n*.

Thus, it has been 30 years since a new lower bound has been well established (as far as this author has discovered). In this paper, we chose to extend Woolbright's elegant proof to accommodate for the *r*-multi Latin square, a generalization of the Latin square, to see if we can establish a lower bound for its maximum transversal.

II. MAIN BODY

A. Values of *r* that Guarantee a Transversal

We will first consider Question 1 which asks how large must r be in order to guarantee the existence of a transversal. We begin with a somewhat obvious, but powerful lemma that will allow us to easily prove further statements

Lemma 1.1: *Any subset of size $q \le r$ cells from a diagonal collection in an r-multi Latin square of order n contains a partial transversal of length q .*

Proof: Suppose *R* is an *r*-multi-Latin square of order *n*. Choose any diagonal collection *D* in *R*. Now, choose any q cells from *D*. We label the cells $C_1, C_2, ..., C_{q-1}, C_q$ in no particular order. By definition, each cell is in a unique column and a unique row. Without losing generality, we will say each cell C_k is in row i_k and column j_k . From C_1 , we choose any symbol s_1 . Clearly $\{(i_1, j_1, s_1)\}$ is partial transversal of length 1. Now if $r \ge 2$, from C_2 , choose any symbol $s_2 \ne s_1$. This symbol s_2 must exist as there are $r \ge 2$ distinct symbols within each cell. Thus, we have a partial transversal of length 2, namely $\{(i_1, j_1, s_1), (i_2, j_2, s_2)\}$. If $r \ge 3$, for induction, suppose $C_1, C_2, ..., C_{k-1}, C_k$ $(k < q)$ contains a partial transversal of length k , namely $\{(i_x, j_x, s_x) | 1 \le x \le k\}$. Now, from C_{k+1} , choose any symbol $s_{k+1} \notin \{s_1, s_2, ..., s_{k-1}, s_k\}$. This symbol s_{k+1} must exist as there are $r > k$ distinct symbols within each cell. Thus, we have a partial transversal of length $k+1$, namely $\{(i_x, j_x, s_x) | 1 \le x \le k+1\}$. Hence, by induction, *R* contains a partial transversal of length $q \le r$, namely $\{(i_x, j_x, s_x) | 1 \le x \le q\}$. □

Theorem 1.2: *Every r-multi Latin square of order* $n > r$ *contains a partial transversal of size* r.

Proof: Let P be the set of cells along the main diagonal. Choose any *r* of these cells. By Lemma 1.1, *P* contains a partial transversal of length *r*. □

Theorem 1.3: *Every r-multi Latin square of order* $n \le r$ *contains a transversal.*

Proof: Let T be the set of cells along the main diagonal. By Lemma 1.1, *P* contains a partial transversal of length *n*. By definition, this partial transversal is a transversal. □

The above theorems allow us to easily establish a lower bound on the length of a partial transversal as stated below in *Corollary 1.4*.

Corollary 1.4: *Every r-multi Latin square of order n contains a partial transversal of size min{r, n}.*□

Now, by implementing a simple contradictory argument, we can decrease *r* to $n-1$ and still obtain a transversal as shown below.

Theorem 1.5: *Every r-multi Latin square of order n where* $r = n-1$ *contains a transversal.*

Proof: Suppose *R* is an *r*-multi-Latin square of order *n* where $r = n-1$. We label the symbols in the first row of *R* in sequential order as follows. The contents of cell $(1, j) = \{(j-1)\, r + k \mid k = 1,2,\ldots,r\}$ for all $j = 1,2,\ldots,n$.

$1,\ 2,\ 3,\ \ldots\ r$	$r+1,\ r+2,\ \ldots\ 2r$	$2r+1,\ 2r+2,\ \ldots\ 3r$	$\cdots$	$(n-1)r+1,\ (n-1)r+2,\ \ldots\ nr$
...				

Now, for the purpose of contradiction, suppose that *R* does not contain a transversal. The only way this can be achieved is if every cell of any given diagonal collection in *R* contains the exact same contents. Otherwise, a transversal exists. Now, note cell (2, 1) is in 2 distinct diagonal collections with both (1, 2) and (1, 3). But (1, 2) and (1, 3) must contain different contents. Thus, every cell of any given diagonal collection of cells cannot have the same contents. Hence, *R* must contain a transversal. □

Once we decrement *r* again to $n-2$, the existence of a transversal becomes a bit more difficult to prove as shown below.

Theorem 1.6: *Every r-multi Latin square of order n where* $r=n-2$ *contains a transversal.*

Proof: Suppose *R* is an *r*-multi-Latin square of order *n* where $r=n-2$.

Case $r=1$: It is well known that any 1-multi Latin square of order 3 (or Latin square of order 3) contains a transversal.

Case $r=2$: We begin by showing that if 4 distinct symbols lie within 2 cells in a diagonal collection, a transversal must exist. Without loss of generality, we may assume the diagonal collection lies along the main diagonal and the cells may be placed in any order. We divide this case into the following 4 subcases.

Subcase 1: At least 5 unique symbols lie in the diagonal collection.

Thus, the symbols in the diagonal collection must take on a form isomorphic to

1 2			
	3 4		
		5 *A*	
			B *C*

To obtain a transversal, we first choose the ordered triple $(3,3,5)$ which refers to symbol 5 in row 3 and column 3. Next, symbol *B* or symbol *C* must be different from symbol 5. Call this symbol *X* and choose the ordered triple $(4,4,X)$. Next, symbol 1 or 2 must be different from symbol *X*. Call this symbol *Y* and choose the ordered triple $(1,1,Y)$. Next, symbol 3 or 4 must be different from symbol *X*. Call this symbol *Z* and choose the ordered triple $(2,2,Z)$. Hence, we have the transversal $\{(1,1,Y),(2,2,Z),(3,3,5),(4,4,X)\}$.

Subcase 2: Exactly 3 cells within the diagonal collection contain the same symbols.

Thus, the diagonal collection must take on a form isomorphic to

1 2			
	1 2		
		1 2	A B
		C D	3 4

Note that $A,B,C,D \notin \{1,2\}$. Also, note that either symbol *C* or symbol *D* must be different symbol *A*. We will call this symbol *X*. Hence, we have the transversal $\{(1,1,1),(2,2,2),(3,4,A),(4,3,X)\}$.

Subcase 3: Exactly 2 cells within the diagonal collection contain the same symbols and Subcase 1 does not apply.

Thus, the diagonal collection must take on a form isomorphic to

1 2			
	1 2		
		3 4	
			A B

Note that $A,B \in \{1,2,3,4\}$ or else *Subcase* 1 applies. In addition $\{A,B\} \neq \{1,2\}$ or else more than exactly 2 cells contain the same symbols. So at least one of the symbols *A* or *B* must be different from symbol 1 and symbol 2. Call this symbol X so $X \notin \{1,2\}$. One of the symbols 3 or 4 must not be equal to X. Call this symbol Y. Hence, we have the transversal

$\{(1,1,1),(2,2,2),(3,3,Y),(4,4,X)\}$.

Subcase 4: All other cases that do not follow Subcases 1, 2, or 3.

Thus, the diagonal collection must take on a form isomorphic to

1 2			
	3 4		
		A B	
			C D

Note $A,B,C,D \in \{1,2,3,4\}$ or else *Subcase* 1 applies. In addition, $\{A,B\},\{C,D\} \notin \{\{1,2\},\{3,4\}\}$ or else *Subcase* 2 or *Subcase* 3 applies. So if we examine the set $\{A,B\}$, exactly one of the symbols, call it X, must be either 1 or 2. Let $\bar{X} = \{1,2\} \setminus X$. Similarly, exactly one of the symbols in the set $\{C,D\}$, call it Y, must be either 3 or 4. Let $\bar{Y} = \{3,4\} \setminus Y$. Hence, we have the transversal $\{(1,1,\bar{X}),(2,2,\bar{Y}),(3,3,X),(4,4,Y)\}$.

Now that we have shown a transversal exists if 4 distinct symbols lie within 2 cells of a diagonal collection, next we will show R contains such a diagonal collection. We label the symbols in the first row in sequential order as follows.

1 2	3 4	5 6	7 8
$\uparrow$			
X Y			
$\downarrow$			

By definition, the first column of R must contain exactly 1 occurrence of both symbols 3 and 4. Thus, at least one (and at most 2) of the cells $(1,2)$, $(1,3)$, or $(1,4)$ contains symbols $X,Y \notin \{3,4\}$. This cell combined with $(2,1)$ (containing symbols 3, 4) is a subset of cells in a diagonal collection. Hence, 4 distinct symbols lie within 2 cells in a diagonal collection, and *R* contains a transversal.

Case $r \geq 3$: Examine the main diagonal. By Lemma 1.1, cells $(1,1)$, $(2,2)$, $\ldots$, (r,r) must contain a partial transversal *P* of length *r*. Now, examine the lower right 2 × 2 block of cells in *R*. This block of cells must contain exactly $2r$ pairwise distinct symbols in each column and each row. So one of the 2 diagonal collections of the 2 × 2 block must introduce at least $\frac{2r-r}{2} = \frac{r}{2}$ (if *r* is even) or $\frac{2r-r+1}{2} = \frac{r+1}{2}$ (if *r* is odd) symbols not in *P*. Specifically, since $r \geq 3$, a diagonal collection of the 2 × 2 block exists that contains at least 2 symbols (in each cell) not in *P*. Hence, 1 unique symbol from each of these 2 cells combined with *P* form a transversal. □

I do believe r could still be reduced significantly and we could still obtain a transversal. I say this as the above proofs do not take advantage of every constraint that is placed on each cell by every other cell. Even if two cells are not in the same row or column, constraints are still imposed as a neighbor in one cell's row is also a neighbor in the cell's column and vice-versa.

B. Generalizing Woolbright's Proof

Next, we consider Question 2 which asks if we can generalize David E. Woolbright's proof in order to establish a lower bound for the maximal length of a partial transversal in any *r*-multi Latin square of order *n.* Woolbright used the proper *n*-coloring of a complete bipartite graph representation to establish a lower bound of $n-\sqrt{n}$ for the maximal length of a partial transversal in any Latin square of order *n*. In order to generalize the proof, we use a proper *nr*-coloring of a bipartite multigraph as *Theorem 2.1* below explains.

Theorem 2.1: *Any r-multi Latin square of order n contains a partial transversal of length at least* $\frac{(1+n+nr)-\sqrt{(1+n+nr)^2-4n^2r}}{2}$.

Proof: Let R be any r-multi Latin square. Note that it is equivalent to a proper nr-coloring of a bipartite multigraph M where exactly r distinctly colored edges connect each unique pair of vertices constructed from exactly 1 vertex from each of the partitions $A=\{a_1,a_2,...,a_{n-1},a_n\}$ and $B=\{b_1,b_2,...,b_{n-1},b_n\}$. Thus, a partial transversal P of size p is equivalent to p distinctly colored parallel edges in M. Let P be a partial transversal in M where the size of P is at a maximum value of p. Let E be the collection of nr edges that correspond to a diagonal collection containing P. Without loss of generality, we will rename the vertices in A and B such that $E=\{(a_i,b_i)\,|\,i\in 1,2,...,n\}$ and that $Q=\{(a_i,b_i)\,|\,i\in 1,2,...,p\}$ is a set of p edges with pairwise distinct colorings. Since P is a partial transversal of maximal size in R, Q is a maximal collection of parallel edges in M with pairwise distinct colorings. Let $C=\{c_1,c_2,...,c_{nr-1},c_{nr}\}$ be nr distinct colors where $\{c_1,c_2,...,c_{nr-p}\}$ are the colors that do not occur in Q.

Let $A_1,A_2,...,A_k$ and $B_1,B_2,...,B_k$ be 2 sequences of sets of vertices in A and B such that:

(1) $A_1=\{a_n,a_{n-1},...,a_{p+1}\}$, and $B_1=\{b_n,b_{n-1},...,b_{p+1}\}$,

(2) $A_i\subset A_{i+1}$ and $B_i\subset B_{i+1}$,

(3) $|A_{i+1}\setminus A_i|=|B_{i+1}\setminus B_i|=n-p$,

(4) the $nr-p$ edges with vertices in $B_{i+1}\setminus B_i$ which are colored c_i all have vertices in A_i ,

(5) and any edge (a,b) such that $a\in A_i$ and $b\in B_1$ is not colored c_j if $i\le j\le nr-p$.

We show that if $A_1, A_2, ..., A_k$ and $B_1, B_2, ..., B_k$ are sequences with the 5 properties above and if $k < nr - p + 1$, then we can always construct sets A_{k+1} and B_{k+1} such that the sequences $A_1, A_2, ..., A_{k+1}$ and $B_1, B_2, ..., B_{k+1}$ also follow the 5 properties above. If $k < nr - p + 1$, construct A_{k+1} and B_{k+1} as follows. Consider the $k(nr - p)$ edges with vertices in A_k which are colored c_k. None of these edges have a vertex in B_1 because of (5) so at least $nr - p$ of these edges must have a vertex in $B \setminus B_k$. Choose these $nr - p$ edges and let $B_{k+1} \setminus B_k$ be the vertices of these edges which are in *B*. Let $A_{k+1} = A_k \cup \{a_i \mid (a_i, b_i) \in Q \text{ and } b_i \in B_{k+1} \setminus B_k\}$. Since these sequences follow properties (1), (2), (3), and (4), we need only to verify property (5). For contradiction, suppose (a,b) is any edge in *Q* with $b \in B_{j+1} \setminus B_j$, $i \le j \le k$, and there is an edge (a^*, b) colored c_j where $a^* \in A_j$. As a consequence, if (a,b) is in *Q* with $b \in B_{j+1} \setminus B_j$, $i \le j \le k$, then there exist edges $(a = x_1, b = y_1), (x_2, y_2), ..., (x_s, y_s)$ in *Q* such that the edges $(x_2, y_1), (x_3, y_2), ..., (x_s, y_{s-1}), (z, y_s)$ are colored $c_j = c_{i_1}, c_{i_2}, ..., c_{i_s}$ where $j = i_1 > i_2 > ... > i_s \ge 1$ and $z \in A_1$. Now, suppose $e = (a, c)$ is an edge colored c_m, $k + 1 \le m \le nr - t,$ with $a \in A_{k+1}$ and $c \in B_1$. By (5), $a \notin A_k$ so $a \in A_{k+1} \setminus A_k$. If (a,b) is the edge in *Q* containing a, then $b \in B_{k+1} \setminus B_k$. Hence, there exist edges $(a = x_1, b = y_1), (x_2, y_2), ..., (x_s, y_s)$ in *Q* such that the edges $(x_2, y_1), (x_3, y_2), ..., (x_s, y_{s-1}), (z, y_s)$ are colored $c_k = c_{i_1}, c_{i_2}, ..., c_{i_s}$ where $k = i_1 > i_2 > ... > i_s \ge 1$ and $z \in A_1$. But if we remove the s edges $(x_1, y_1), (x_2, y_2), ..., (x_s, y_s)$ from *Q* and replace them with the $s + 1$ edges $e = (a, c), (x_2, y_1), (x_3, y_2), ..., (x_s, y_{s-1}), (z, y_s)$, we have a set of $p + 1$ parallel edges with pairwise distinct colorings. This contradicts the supposition that the maximum number of parallel edges with pairwise distinct colorings is p.

Hence, the extended sequences we have constructed follow property (5). Finally, since A_1 and B_1 as defined above follow all 5 stated properties, they can be extended to the sequences $A_1, A_2, ..., A_{nr-p+1}$ and $B_1, B_2, ..., B_{nr-p+1}$ which follow the same 5 properties. Since $|A_{i+1} \setminus A_i| = n-p$, clearly $n \geq (n-p)(nr-p+1)$. Thus, $p^2 - p(1+n+nr) + n^2 r \leq 0$ and

$$\frac{(1+n+nr)-\sqrt{(1+n+nr)^2-4n^2r}}{2} \leq p \leq n < \frac{(1+n+nr)+\sqrt{(1+n+nr)^2-4n^2r}}{2} \text{ . } \square$$

Now, by substituting the values 1, 2, and 3 for *r* in *Theorem 2.1*, we can show Corollaries *2.2*, *2.3*, and *2.4*, respectively.

Corollary 2.2: *Any 1-multi Latin square (or Latin square) of order n contains a partial transversal of length at least* $n+0.5-\sqrt{n+0.25}$.□

Corollary 2.3: *Any 2-multi Latin square of order n contains a partial transversal of length at least* $1.5n+0.5-\sqrt{0.25n^2+1.5n+0.25}$.□

Corollary 2.4: *Any 3-multi Latin square of order n contains a partial transversal of length at least* $2n+0.5-\sqrt{n^2+2n+0.25}$.□

As you can see, the length of a required partial transversal increases each time *r* is increased. *Figure 2* below is a plot comparing these three results to Woolbright's original result as well as the result when *r* = 10,000,000. It appears that as *r* approaches infinity, the lower bound on maximal partial transversal length p approaches n. Also, our result concerning 1-multi Latin Squares (or Latin squares) slightly improves upon Woolbright's original result (as it appears Woolbright elected to drop a term to simplify his result to the more elegant $n-\sqrt{n}$).

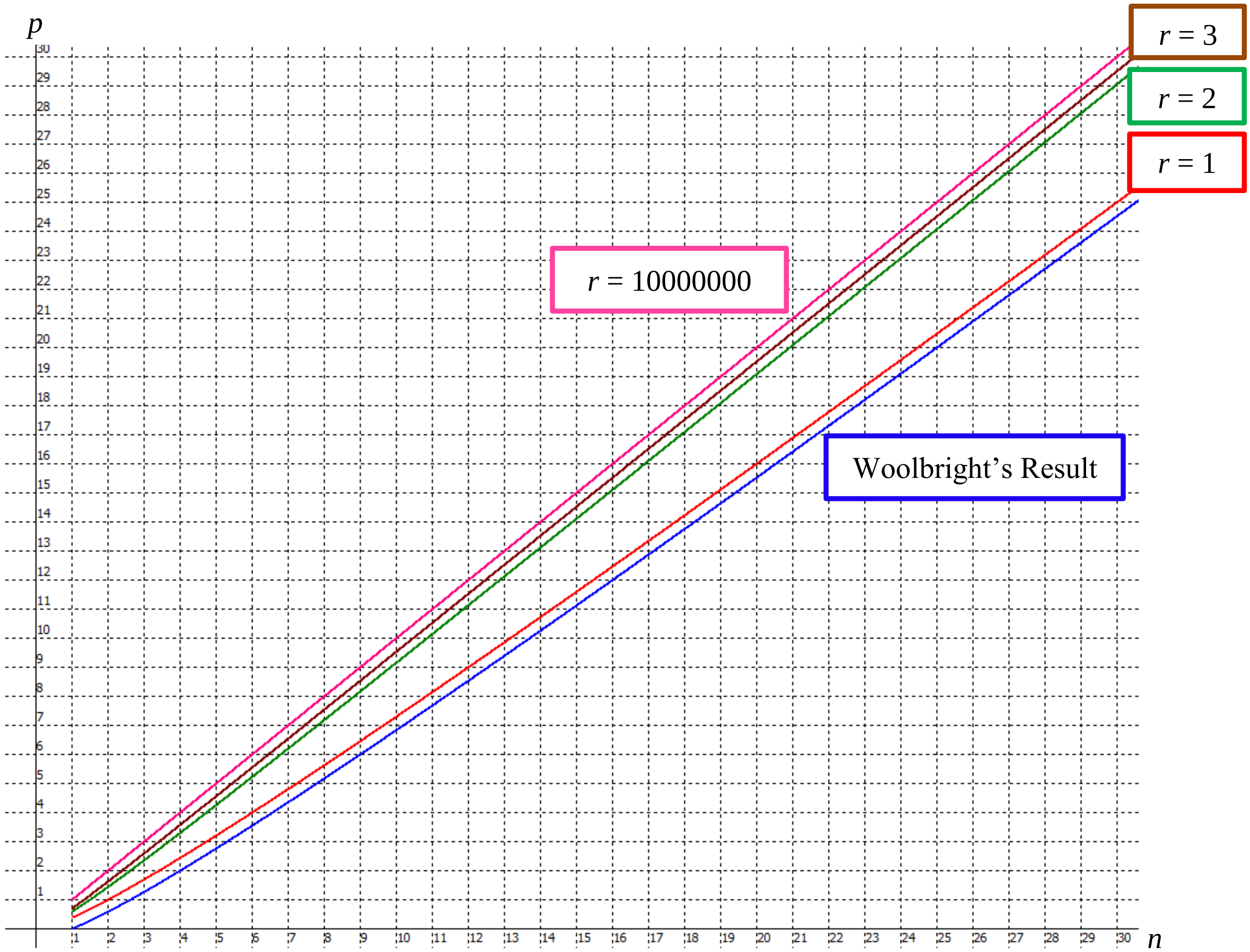

Figure 2. Partial Transversal Length p versus r-multi Latin Square order n.

As speculated, next we will show the result of *Theorem 2.1* is always less than the order n. We start with an obvious statement.

$$n > 0$$
$$\Rightarrow n + n^2 r > n^2 r$$
$$\Rightarrow -2n(2 + 2nr) < -4n^2 r$$
$$\Rightarrow (1 - n + nr)^2 - (1 + n + nr)^2 < -4n^2 r$$
$$\Rightarrow 1 - n + nr < \sqrt{(1 + n + nr)^2 - 4n^2 r}$$
$$\Rightarrow 1 - n + nr - \sqrt{(1 + n + nr)^2 - 4n^2 r} < 0$$
$$\Rightarrow 1 + n + nr - \sqrt{(1 + n + nr)^2 - 4n^2 r} < 2n$$
$$\Rightarrow \frac{1 + n + nr - \sqrt{(1 + n + nr)^2 - 4n^2 r}}{2} < n$$

Thus, the result obtained in *Theorem 2.1* can never guarantee a transversal.

C. Transversals on any *n* Symbols

Finally, we consider Question 3 which asks can we find a transversal on *n* specified symbols if *r* is large enough. Due to the complexity of this question and limited time, we will answer this question for orders $n = 1$ and $n = 2$ only. We leave all other orders as a challenge for the reader.

Theorem 3.1: *An r-multi Latin square of order 1 contains a transversal on any 1 symbol.*

*Proof***:** Let s be any symbol in cell $(1,1)$. Clearly, $\{(1,1,s)\}$ is a transversal. □

Theorem 3.2: *An r-multi Latin square of order 2 does not contain a transversal on any 2 symbols.*

*Proof***:** Let R be an *r*-multi Latin square of order 2. Choose any symbol s_1 from cell $(1,1)$ and any symbol s_2 from cell $(1,2)$. These 2 symbols are not the same since they reside in the same row. In addition, s_1 does not reside in $(1,2)$ or $(2,1)$. Furthermore, s_2 does not reside in $(1,1)$ or $(2,2)$. Hence, a transversal does not exist with symbols s_1 and s_2 .□

III. CONCLUSIONS

A. Summary: Interpretation

In conclusion, we successfully addressed all questions posed to a certain degree. We have shown any *r*-multi Latin square contains a transversal if $r \geq n-2$. However, I feel this could still be reduced dramatically and transversals will still exist. During research, an *r*-multi Latin square (where $r \geq 2$) without a transversal was never discovered and seemed difficult, if not impossible, to create.

We did successfully extend Woolbright's proof to establish a lower bound for the maximal size of a partial transversal in any *r*-multi Latin square. The plot in *Figure 2* shows how this result compares to Woolbright's original result for varying values of *r*. We also showed that the result in *Theorem 2.1* cannot guarantee any transversals as the lower bound on maximal partial transversal length established here is always less than *n*. Consequently, $\min\{n, r\}$, the result shown in *Corollary 1.4*, is a higher and therefore better lower bound on partial transversal length when $r \geq n$.

We also began to answer the question of whether or not transversals exist on any given set of *n* symbols if *r* is large enough. Due to limited time and the complexity of the question, we could only completely answer this question for orders 1 and 2. We did discover that it is true for all values of *r* with an order of 1, and it is false for all values of *r* with an order of 2. The remaining orders remain a mystery.

B. Suggestions for Further Study

Here is a list of questions that remain unanswered and can provide hours of fun research for those ready for a challenge.

1) Can we always find a transversal in an r-multi Latin square on any n symbols if r is large enough? We of course answered this question for the simplistic orders of 1 and 2. What about if $r \geq 3$?

2) By now, we are familiar with Brualdi's and Ryser's conjectures regarding even and odd order Latin squares. Are there any similar conclusions or conjectures we can make regarding the size of partial transversals in even and odd order r-multi Latin squares? What about even and odd r?

3) During research, we discovered creating or finding an r-multi Latin square $(\text{where } r \geq 2)$ is a rather difficult, if not impossible, task. Are there any examples of r-multi Latin squares $(\text{where } r \geq 2)$ without transversals? Can some computer code be written to conduct a systematic search for such an r-multi Latin square?

4) Can we always 'extract' a Latin Square from an r-multi Latin square? That is, can we always strategically choose exactly one symbol from each cell to form a Latin square? If this can be done, by what method, and how many disjoint Latin squares can we obtain? Furthermore, IF we can show that at least 2 orthogonal Latin Squares can be extracted from an r-multi Latin square, then a transversal must exist as mutually orthogonal Latin squares require the presence of disjoint transversals.

5) What is a formula $N(r,n)$ to obtain the number of possible r-multi Latin squares of order n? This question is still also open for traditional Latin squares as well.

6) What are the bounds on the number of transversals in an r-multi Latin squares of order n?

As you can see, many questions remain regarding r-multi Latin squares. We hope the reader accepts the challenge of addressing at least 1 of them.

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QED

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